Caching or pre-fetching? The role of hazard rates.

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joint work with Matias Carrasco and Fernando Paganini

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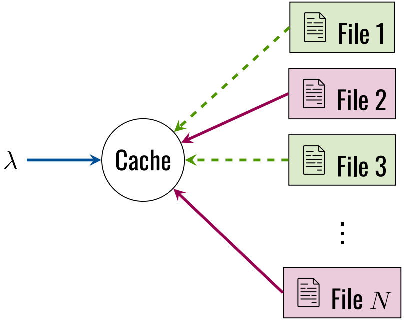
60th Allerton Conference on Computing, Control and Communications – September 2024

The caching problem

- Consider a local memory system that handles items from a catalog of N objects.
- Requests for objects arrive as a random process.
- The memory (cache) can locally store $C < N$ of them.
- If item is in cache, we have a hit. Otherwise, it is a miss.

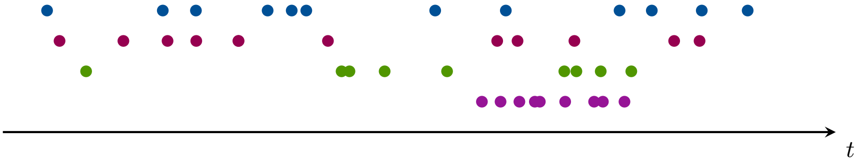


Objective: for a given arrival stream, maximize the steady-state hit rate.

Point process approach

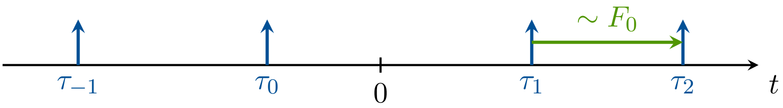
Introduced in [Fofack et al. 2014]

- Assume requests for item i come from a point process of intensity λ_i (popularities).



- At each point in time we must decide which items must be stored locally.

Two important distributions:



- **Inter-arrival distribution:** Typical distance between points...

$$\tau_{k+1} - \tau_k \sim F_0(t), \quad E[\tau_{k+1} - \tau_k] = 1/\lambda$$

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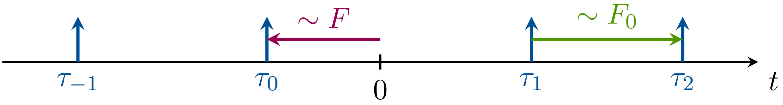

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$$F(t) := \lambda \int_0^t 1 - F_0(s)ds,$$

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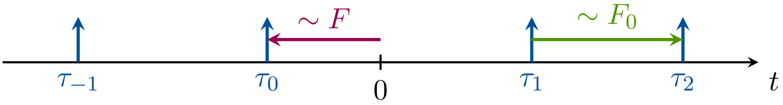

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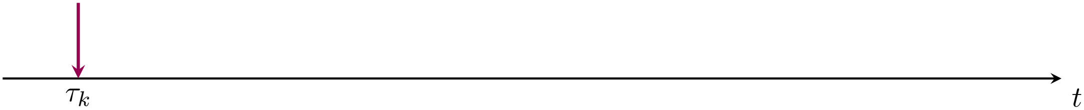
Note: you can formalize this under the Palm probability framework for stationary point processes.

Populating a cache: timer based (TTL) policies

- Upon request arrival for item i, check for presence.



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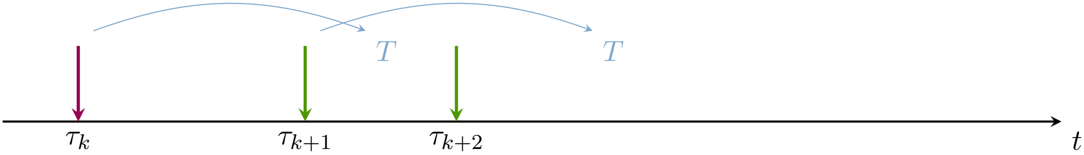

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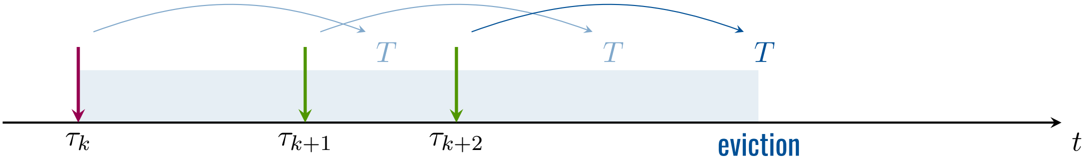

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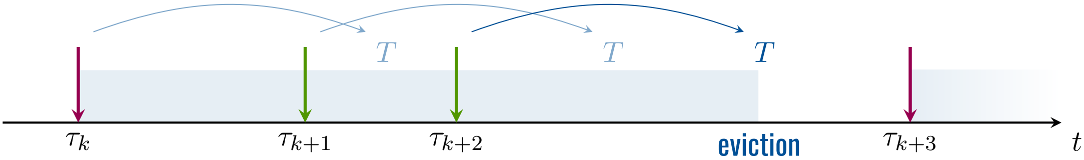

Structure of the optimal caching policy

- The crucial magnitude is the hazard rate of F_0:

$$\eta(t) := \frac{f_0(t)}{1 - F_0(t)}$$

- Likelihood of a request at time t, given the current interval has age t.

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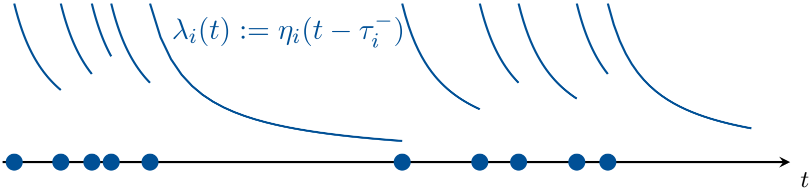
Theorem (F', Rodriguez, Paganini – 2016)

If the $F_0^{(i)}$ have decreasing hazard rates, then the optimal TTL policy satisfies:

$$\eta_i(T_i^*) \geqslant \theta^*,$$

whenever $T_i^* > 0$ (i.e. the item is cached).
Moreover, inequality is strict iff $T_i^* = \infty$ (item always stored).

Decreasing hazard rates corresponds to bursty traffic:



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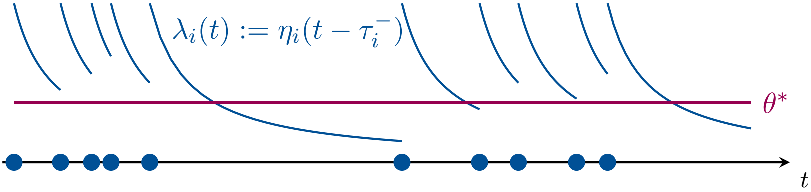

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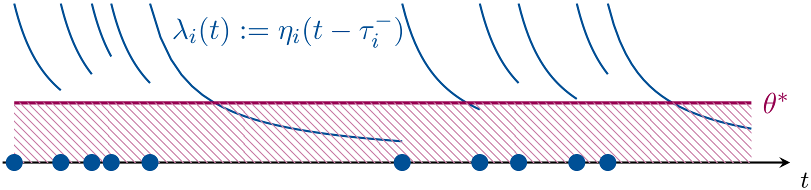

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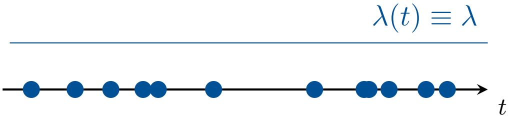
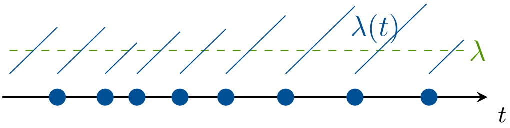
What about other types of traffic?



Constant hazard rate $\rightarrow$ Poisson process.



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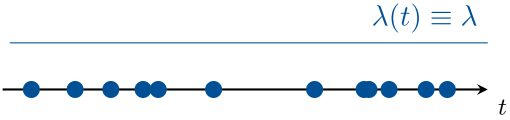

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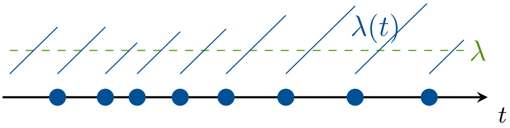

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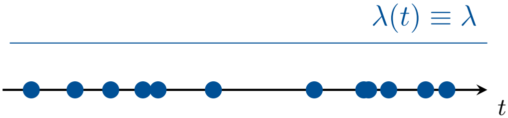

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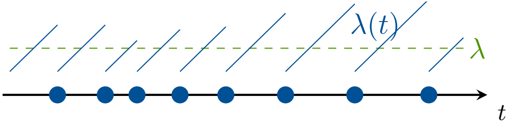

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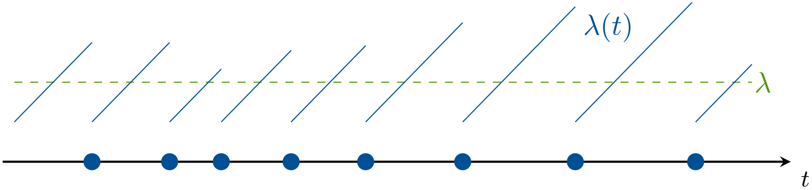
Can we improve upon this?

Thinking about increasing hazard rates...

- Once you have seen a request, it's less likely to see the same item again for a while.



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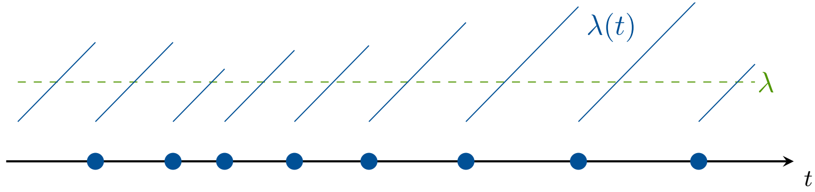

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Key insight

The question now is not how long we should remember something, but instead how long we should forget about it!

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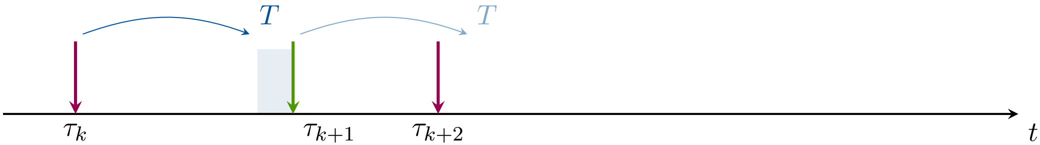

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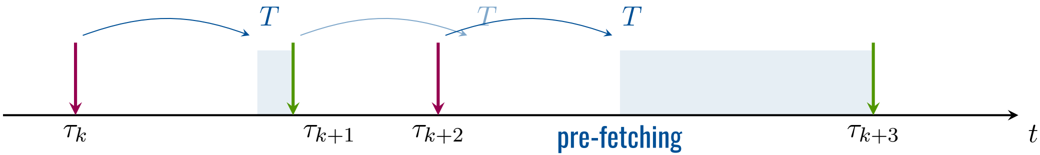

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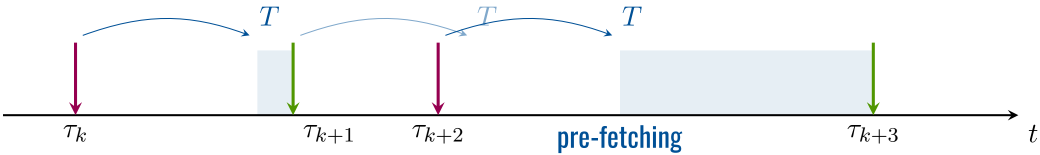
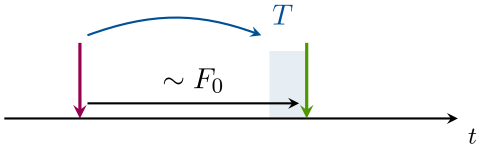


Timer based pre-fetching

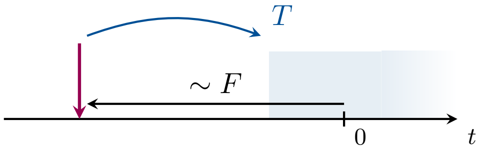
Consider a single item with a timer T and its request process:

Hit probability: next arrival occurs after timer expires.



Hit probability $= 1 - F_0(T)$

Occupation probability: probability that timer has expired by 0 since last arrival.



Avg. occupation $= 1 - F(T)$

Choosing the optimal timers

Requests come from independent sources with intensities λ_i and inter-arrival distribution F_i:

Problem (Optimal pre-fetching policy)

Choose timers $T_i \geqslant 0$ such that:

$$\max_{T_i \geqslant 0} \sum_i \lambda_i (1 - F_0^{(i)}(T_i))$$

subject to:

$$\sum_i (1 - F^{(i)}(T_i)) \leqslant C$$

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subject to:

$$\sum_i F^{(i)}(T_i) \geqslant N - C$$

- Apply the change of variables $u_i = F^{(i)}(T_i)$.
- Note that u_i is the probability of not being stored.
- The problem becomes:

$$\min_{u_i \in [0,1]} \sum_i \lambda_i \left[F_0^{(i)} \circ (F^{(i)})^{-1} \right] (u_i)$$

subject to:

$$\sum_i u_i \geqslant N - C$$

- Objective gradient:

$$\frac{\partial}{\partial u_i}\lambda_i F^{(i)} \circ (F^{(i)})^{-1}(u_i) = \frac{\lambda_i f_0^{(i)}((F^{(i)})^{-1}(u_i))}{\lambda_i \left(1 - F_0^{(i)}((F^{(i)})^{-1}(u_i))\right)} = \eta_i((F^{(i)})^{-1}(u_i))$$

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- Increasing! $\rightarrow$ Proper convex optimization problem.
- Lagrangian duality:

$$\begin{aligned}\mathcal{L}(u,\theta) &= \sum_{i=1}^{N} \lambda_i F_0^{(i)} \left((F^{(i)})^{-1}(u_i)\right) + \theta \left(N - C - \sum_{i=1}^{N} u_i\right) \\ &= \sum_{i=1}^{N} \left[\lambda_i F_0^{(i)} \left((F^{(i)})^{-1}(u_i)\right) - \theta u_i\right] + \theta(N - C).\end{aligned}$$

Optimal Timer-based pre-fetching

Theorem

If the $F_0^{(i)}$ satisfy the IHR property, there exists a unique threshold $\theta^* \geqslant 0$ such that the optimal timers satisfy:

$$\eta_i(T_i^*) \geqslant \theta^*,$$

whenever $T_i^* < \infty$ (pre-fetching).
The inequality is strict if and only if $T_i^* = 0$, i.e. the content is always stored.

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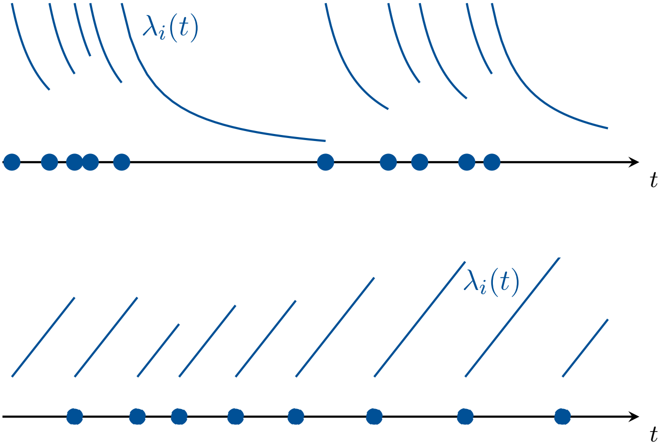
Remark: The policy is also a threshold policy, like the caching case.

A tale of two policies...



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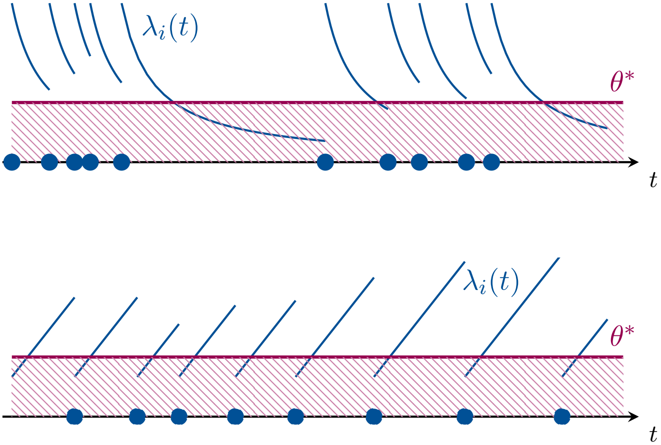
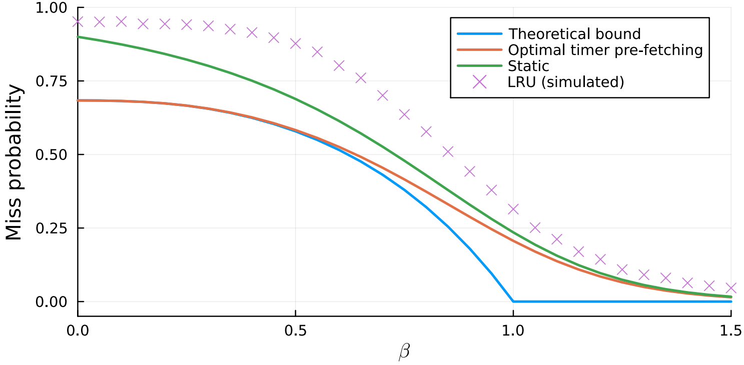


Both policies are just the same policy!

- Keep a hazard rate threshold θ for storing a content
- Compute θ^* such that avg. memory occupation is C.

Simulation example

Erlang ($k = 5$) inter-arrival times, Zipf $\propto n^{-\beta}$ popularities, varying β, $N = 10000$, $C = 1000$.



- Optimal pre-fetching improves over the static policy.
- Classical caching (e.g. LRU) is a very bad idea for regular traffic.

Theorem

Consider a family of local memory systems, indexed by N, with inter-request times coming from a common scale family, and memory size $C_N = cN$. Then the hazard rate threshold θ_N^* verifies:

$$\theta_N \xrightarrow[N\to\infty]{} \theta^*,$$

where θ^* is the solution of:

$$G_\infty(\theta^*) = 1 - c,$$

and G_∞ depends on the age distribution F and the popularity distribution L.

Main results

Asymptotic miss probability

Theorem

Consider a family of local memory systems as before, then the miss probability for system N, M_N, satisfies:

$$M_N \xrightarrow[N\to\infty]{} \frac{\displaystyle\int_0^\infty \lambda G_0(\theta^*/\lambda)L(d\lambda)}{\displaystyle\int_0^\infty \lambda L(d\lambda)},$$

with θ^* as before, L is the distribution of popularities, and G_0 depends on the inter-arrival distribution F_0.

Universal optimality

Theorem (In preparation – check ArXiv soon)

Under the above assumptions, the (hard to compute) optimal causal policy converges to a fixed threshold policy with the same limit threshold θ^*.

Therefore, timer policies give a universal asymptotic upper bound on caching/pre-fetching performance.

Final remarks

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- We identified the hazard rate as the crucial indicator of regularity, and devised a new policy for IHR, that is also asymptotically optimal among all causal policies.

Final remarks

- You have to know your traffic before deciding on caching or pre-fetching!

- Classical caching is not well suited to regular traffic.

- We identified the hazard rate as the crucial indicator of regularity, and devised a new policy for IHR, that is also asymptotically optimal among all causal policies.

- A lot of open questions, in particular:
  - How we can learn the hazard rates online?
  - How we can estimate the appropriate threshold?
  - What about mixtures of IHR and DHR traffic?

Thank you!

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